

L-functions

(PARI-GP version 2.15.3)

Characters

A character on the abelian group $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$, e.g. from **znstar**(**q**,1) $\leftrightarrow (\mathbf{Z}/q \mathbf{Z})^*$ or **bnrinit** $\leftrightarrow \text{Cl}_{\mathfrak{f}}(K)$, is coded by $\chi = [c_1, \dots, c_k]$ such that $\chi(g_j) = e(c_j/d_j)$. Our L -functions consider the attached *primitive* character.

Dirichlet characters $\chi_q(m, \cdot)$ in Conrey labelling system are alternatively concisely coded by **Mod**(**m**,**q**). Finally, a quadratic character (D/\cdot) can also be coded by the integer D .

L-function Constructors

An **Ldata** is a GP structure describing the functional equation for $L(s) = \sum_{n \geq 1} a_n n^{-s}$.

- Dirichlet coefficients given by closure $a : N \mapsto [a_1, \dots, a_N]$.
- Dirichlet coefficients $a^*(n)$ for dual L -function L^* .
- Euler factor $A = [a_1, \dots, a_d]$ for $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$,
- classical weight k (values at s and $k - s$ are related),
- conductor N , $\Lambda(s) = N^{s/2} \gamma_A(s)$,
- root number ε ; $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$.
- polar part: list of $[\beta, P_{\beta}(x)]$.

An **Linit** is a GP structure containing an **Ldata** L and an evaluation *domain* fixing a maximal order of derivation m and bit accuracy (**realbitprecision**), together with complex ranges

- for L -function: $R = [c, w, h]$ (coding $|\Re z - c| \leq w$, $|\Im z| \leq h$); or $R = [w, h]$ (for $c = k/2$); or $R = [h]$ (for $c = k/2$, $w = 0$).
- for θ -function: $T = [\rho, \alpha]$ (for $|t| \geq \rho$, $|\arg t| \leq \alpha$); or $T = \rho$ (for $\alpha = 0$).

Ldata constructors

| | |
|--------------------------------------------------------------------------|-------------------------------------------------------------------|
| Riemann ζ | lfuncreate (1) |
| Dirichlet for quadratic char. (D/\cdot) | lfuncreate (D) |
| Dirichlet series $L(\chi_q(m, \cdot), s)$ | lfuncreate (Mod (m , q)) |
| Dedekind ζ_K , $K = \mathbf{Q}[x]/(T)$ | lfuncreate (<i>bnf</i>), lfuncreate (T) |
| Hecke for $\chi \bmod \mathfrak{f}$ | lfuncreate (<i>bnr</i> , χ) |
| Artin L -function | lfunartin (<i>nf</i> , <i>gal</i> , M , n) |
| Lattice Θ -function | lfunqf (Q) |
| From eigenform F | lfunmf (F) |
| Quotients of Dedekind $\eta: \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$ | lfunetaquo (M) |
| $L(E, s)$, E elliptic curve | E = ellinit (...) |
| $L(\text{Sym}^m E, s)$, E elliptic curve | lfunsympow (E , m) |
| Genus 2 curve, $y^2 + xQ = P$ | lfungenus2 ($[P, Q]$) |
| Hypergeometric motive $H(a, b; t)$ | lfunhgm (hgmin (a , b), t) |

| | |
|------------------------------|-------------------------------|
| dual L function \hat{L} | lfundual (L) |
| $L_1 \cdot L_2$ | lfunmul (L_1, L_2) |
| L_1/L_2 | fundiv (L_1, L_2) |
| $L(s - d)$ | funshift (L, d) |
| $L(s) \cdot L(s - d)$ | funshift ($L, d, 1$) |
| twist by Dirichlet character | funtwist (L, χ) |

| | |
|-------------------------------------|-------------------------------------------------------------------------|
| low-level constructor | lfuncreate ($[a, a^*, A, k, N, \textit{eps}, \textit{poles}]$) |
| check functional equation (at t) | funcheckfeq ($L, \{t\}$) |
| parameters $[N, k, A]$ | funparams (L) |

Linit constructors

| | |
|--------------------------------------------------|---------------------------------------------------|
| initialize for L | lfuninit ($L, R, \{m = 0\}$) |
| initialize for θ | funthetainit ($L, \{T = 1\}, \{m = 0\}$) |
| cost of lfuninit | funcost ($L, R, \{m = 0\}$) |
| cost of funthetainit | funthetacost ($L, T, \{m = 0\}$) |
| Dedekind ζ_L , L abelian over a subfield | funabelianreinit |

L-functions

L is an **Ldata** or an **Linit** (more efficient for many values).

Evaluate

| | |
|--------------------|----------------------------------------|
| $L^{(k)}(s)$ | lfun ($L, s, \{k = 0\}$) |
| $\Lambda^{(k)}(s)$ | funlambda ($L, s, \{k = 0\}$) |
| $\theta^{(k)}(t)$ | funtheta ($L, t, \{k = 0\}$) |

generalized Hardy Z -function at t **funhardy**(L, t)

Zeros

| | |
|------------------------------------------|-----------------------------------------|
| order of zero at $s = k/2$ | funorderzero ($L, \{m = -1\}$) |
| zeros $s = k/2 + it$, $0 \leq t \leq T$ | funzeros ($L, T, \{h\}$) |

Dirichlet series and functional equation

| | |
|--------------------------|-----------------------------|
| $[a_n: 1 \leq n \leq N]$ | funan (L, N) |
| Euler factor at p | fun euler (L, p) |
| conductor N of L | funconductor (L) |
| root number and residues | funrootres (L) |

G-functions

Attached to inverse Mellin transform for $\gamma_A(s)$, $A = [a_1, \dots, a_d]$.
 initialize for G attached to A **gammamellinin**(A)
 $G^{(k)}(t)$ **gammamellinin**($G, t, \{k = 0\}$)
 asymp. expansion of $G^{(k)}(t)$ **gammamellinin**($A, n, \{k = 0\}$)

Hypergeometric motives (HGM)

Hypergeometric templates

Below, H denotes an hypergeometric template from **hgminit**.
 HGM template from $A = (\alpha_j)$, $B = (\beta_k)$ **hgminit**($A, \{B\}$)
 ...from cyclotomic parameters D, E **hgminit**($D, \{E\}$)
 ...from gamma vector **hgminit**(G)
 α and β parameters for H **hgmalph**(H)
 cyclotomic parameters (D, E) of H **hgmcyclo**(H)
 ...for all H of degree n **hgmbdegree**(n)
 gamma vector for H **hgmgamma**(H)
 twist A and B by $1/2$ **hgmtwist**(H)
 is H symmetrical at $t = 1$? **hgmissymmetrical**(H)
 parameters $[d, w, [P, T], M]$ for H **hgmparams**(H)

L-function

Let L be the L -function attached to the hypergeometric motive (H, t).

| | |
|-----------------------------------------|------------------------------------------|
| coefficient a_n of L | hgcoef (H, t, n) |
| coefficients $[a_1, \dots, a_n]$ of L | hgcoefs (H, t, n) |
| Euler factor at p | hgmeulerfactor (H, t, p) |
| ...and valuation of local conductor | hgmeulerfactor ($H, t, p, \&e$) |
| return L as an Ldata | funhgm (H, t) |

Based on an earlier version by Joseph H. Silverman

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