L-functions

(PARI-GP version 2.15.3)

Characters

A character on the abelian group $G = \sum_{j\le k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$, e.g. from `znstar(q,1)` $\leftrightarrow (\mathbf{Z}/q\mathbf{Z})^*$ or `bnrinit` $\leftrightarrow \mathrm{Cl}_{\mathfrak{f}}(K)$, is coded by $\chi = [c_1, \dots, c_k]$ such that $\chi(g_j) = e(c_j/d_j)$. Our L-functions consider the attached *primitive* character.

Dirichlet characters $\chi_q(m,\cdot)$ in Conrey labelling system are alternatively concisely coded by `Mod(m,q)`. Finally, a quadratic character $(D/\cdot)$ can also be coded by the integer D.

L-function Constructors

An `Ldata` is a GP structure describing the functional equation for $L(s) = \sum_{n\ge1} a_n n^{-s}$.

- Dirichlet coefficients given by closure $a : N \mapsto [a_1, \dots, a_N]$.
- Dirichlet coefficients $a^*(n)$ for dual L-function L^*.
- Euler factor $A = [a_1, \dots, a_d]$ for $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$,
- classical weight k (values at s and $k - s$ are related),
- conductor N, $\Lambda(s) = N^{s/2}\gamma_A(s)$,
- root number ε; $\Lambda(a, k - s) = \varepsilon\Lambda(a^*, s)$.
- polar part: list of $[\beta, P_\beta(x)]$.

An `Linit` is a GP structure containing an `Ldata` L and an evaluation *domain* fixing a maximal order of derivation m and bit accuracy (`realbitprecision`), together with complex ranges

- for L-function: $R = [c, w, h]$ (coding $|\Re z - c| \le w$, $|\Im z| \le h$); or $R = [w, h]$ (for $c = k/2$); or $R = [h]$ (for $c = k/2$, $w = 0$).
- for θ-function: $T = [\rho, \alpha]$ (for $|t| \ge \rho$, $|\arg t| \le \alpha$); or $T = \rho$ (for $\alpha = 0$).

Ldata constructors

Riemann ζ	`lfuncreate`(1)
Dirichlet for quadratic char. $(D/\cdot)$	`lfuncreate`(D)
Dirichlet series $L(\chi_q(m,\cdot), s)$	`lfuncreate(Mod(m,q))`
Dedekind ζ_K, $K = \mathbf{Q}[x]/(T)$	`lfuncreate`(bnf), `lfuncreate`(T)
Hecke for χ mod $\mathfrak{f}$	`lfuncreate`$([bnr, \chi])$
Artin L-function	`lfunartin`(nf, gal, M, n)
Lattice Θ-function	`lfunqf`(Q)
From eigenform F	`lfunmf`(F)
Quotients of Dedekind η: $\prod_i \eta(m_{i,1}\cdot\tau)^{m_{i,2}}$	`lfunetaquo`(M)
$L(E, s)$, E elliptic curve	`E = ellinit(...)`
$L(Sym^m E, s)$, E elliptic curve	`lfunsympow(E, m)`
Genus 2 curve, $y^2 + xQ = P$	`lfungenus2`$([P, Q])$
Hypergeometric motive $H(a, b; t)$	`lfunhgm(hgminit(a,b), t)`

dual L function $\hat{L}$	`lfundual`(L)
$L_1 \cdot L_2$	`lfunmul`(L_1, L_2)
L_1/L_2	`lfundiv`(L_1, L_2)
$L(s - d)$	`lfunshift`(L, d)
$L(s)\cdot L(s - d)$	`lfunshift`$(L, d, 1)$
twist by Dirichlet character	`lfuntwist`(L, χ)

low-level constructor	`lfuncreate`$([a, a^*, A, k, N, eps, poles])$
check functional equation (at t)	`lfuncheckfeq`$(L, \{t\})$
parameters $[N, k, A]$	`lfunparams`(L)

Linit constructors

initialize for L	`lfuninit`$(L, R, \{m = 0\})$
initialize for θ	`lfunthetainit`$(L, \{T = 1\}, \{m = 0\})$
cost of `lfuninit`	`lfuncost`$(L, R, \{m = 0\})$
cost of `lfunthetainit`	`lfunthetacost`$(L, T, \{m = 0\})$
Dedekind ζ_L, L abelian over a subfield	`lfunabelianrelinit`

L-functions

L is an `Ldata` or an `Linit` (more efficient for many values).

Evaluate

$L^{(k)}(s)$	`lfun`$(L, s, \{k = 0\})$
$\Lambda^{(k)}(s)$	`lfunlambda`$(L, s, \{k = 0\})$
$\theta^{(k)}(t)$	`lfuntheta`$(L, t, \{k = 0\})$
generalized Hardy Z-function at t	`lfunhardy`(L, t)

Zeros

order of zero at $s = k/2$	`lfunorderzero`$(L, \{m = -1\})$
zeros $s = k/2 + it$, $0 \le t \le T$	`lfunzeros`$(L, T, \{h\})$

Dirichlet series and functional equation

$[a_n : 1 \le n \le N]$	`lfunan`(L, N)
Euler factor at p	`lfuneuler`(L, p)
conductor N of L	`lfunconductor`(L)
root number and residues	`lfunrootres`(L)

G-functions

Attached to inverse Mellin transform for $\gamma_A(s)$, $A = [a_1, \dots, a_d]$.

initialize for G attached to A	`gammamellininvinit`(A)
$G^{(k)}(t)$	`gammamellininv`$(G, t, \{k = 0\})$
asymp. expansion of $G^{(k)}(t)$	`gammamellininvasymp`$(A, n, \{k = 0\})$

Hypergeometric motives (HGM)

Hypergeometric templates

Below, H denotes an hypergeometric template from `hgminit`.

HGM template from $A = (\alpha_j)$, $B = (\beta_k)$	`hgminit`$(A, \{B\})$
...from cyclotomic parameters D, E	`hgminit`$(D, \{E\})$
...from gamma vector	`hgminit`(G)
α and β parameters for H	`hgmalpha`(H)
cyclotomic parameters (D, E) of H	`hgmcyclo`(H)
...for all H of degree n	`hgmbydegree`(n)
gamma vector for H	`hgmgamma`(H)
twist A and B by $1/2$	`hgmtwist`(H)
is H symmetrical at $t = 1$?	`hgmissymmetrical`(H)
parameters $[d, w, [P, T], M]$ for H	`hgmparams`(H)

L-function

Let L be the L-function attached to the hypergeometric motive (H, t).

coefficient a_n of L	`hgmcoef`(H, t, n)
coefficients $[a_1, \dots, a_n]$ of L	`hgmcoefs`(H, t, n)
Euler factor at p	`hgmeulerfactor`(H, t, p)
...and valuation of local conductor	`hgmeulerfactor`$(H, t, p, \&e)$
return L as an `Ldata`	`lfunhgm`(H, t)

Based on an earlier version by Joseph H. Silverman

Send comments and corrections to ⟨Karim.Belabas@math.u-bordeaux.fr⟩